Rigorous bounds and the replica method for products of random matrices

1986 J. Phys. A: Math. Gen. 19 L1145

(http://iopscience.iop.org/0305-4470/19/18/006)

J. Phys. A: Math. Gen. 19 (1986) L1145–L1152. Printed in Great Britain

LETTER TO THE EDITOR

Rigorous bounds and the replica method for products of random matrices

J P Bouchaud†, A Georges‡, D Hansel§, P Le Doussal‡ and J M Maillard||

† Laboratoire de Spectroscopie Hertzienne de l'Ecole Normale Supérieure, CNRS, 24 rue Lhomond, 75231 Paris Cedex, France
‡ Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, CNRS, associé à l'Université de Paris-Sud, 24 rue Lhomond, 75231 Paris Cedex, France
§ Centre de Physique Théorique de l'Ecole Polytechnique, Groupe de Recherche du CNRS no 48, Route de Saclay, 91128 Palaiseau Cedex, France
|| Laboratoire de Physique Théorique et Hautes Energies, associé au CNRS et l'Université Pierre et Marie Curie, 4 place Jussieu, 75230 Paris Cedex, France

Received 16 January 1986

Abstract. We first develop a method to obtain rigorous bounds for the Lyapounov exponent of products of a random matrix. When applied to a class of 1D problems, including localisation, it reproduces the correct scaling behaviour at the band edge and gives very good approximations of the prefactors. We then study analytically the successive moments of the distribution law for the trace of the random matrix product within the whole energy band. The band centre anomaly is found to affect the whole statistics of the problem and the exact anomalous value of the Lyapounov exponent is recovered through the replica trick.

A direct approach to random matrix products through the calculation of the successive moments of their trace allows to obtain three different kinds of information.

(i) The second moment gives an upper bound on the Lyapounov exponent (LE) of the problem.

(ii) Analytical continuation of the moments towards $q \to 0$, when possible, gives the value of this LE.

(iii) Finally, the whole set of moments allows us to discuss the statistics of this trace through its distribution function.

Point (i) (corresponding to an annealed approximation in disordered spin problems) has been initiated in the review paper of Ishii (1973) but has not been extensively used since. Moreover, the 'replica trick' (point (ii)) has very seldom been used in actual calculations in the context of random matrices (see Kirkman and Pendry 1984).

The first aim of this letter is to develop point (i) for the study of some products of random matrices providing both rigorous upper bounds and approximations of the Lyapounov exponents (LE). As a test of the efficiency of these methods, we have considered the one-dimensional discretised localisation problem (Halperin 1965, Thouless 1972, 1974, Ishii 1973, Derrida and Gardner 1984). For this problem, we give an upper bound on the LE which reproduces the exactly known $\sigma^{1/3}$ scaling behaviour at band edge.

More generally, we have considered the $n \times n$ random matrices associated with the discretisation of the 1D differential equation

$$\psi^{(n)}(x) = \varepsilon(x)\psi(x) \tag{1}$$

where $\psi^{(n)}$ is the nth derivative of ψ and $\varepsilon(x)$ a centred white noise of variance σ.

We obtain the following bound on the largest LE when $\sigma \to 0$:

$$\lambda \leqslant \left(\frac{(2n-2)!}{(n-1)!^2}\sigma\right)^{1/2n-1}. \tag{2}$$

This $1/(2n-1)$ scaling behaviour can also be predicted for instance using a renormalisation argument (Bouchaud and Le Doussal 1986), while the amplitude is unknown (except for $n=2$). A numerical analysis furthermore indicates that the upper bound (2) reproduces the exact amplitude in the limit $n \to \infty$.

We finally turn to points (ii) and (iii) for the one-dimensional localisation problem and develop a simple method to calculate the replica moments for low disorder, allowing us in particular to apply the 'replica trick' even at the anomalous band centre (Kappus and Wegner 1981).

Let us consider $n \times n$ matrices $M(\varepsilon)$ depending on a random parameter ε of mean value μ and variance σ. One is interested in the real part of the Lyapounov exponent of the product $\Pi\, M(\varepsilon_i)$

$$\lambda = \lim_{N\to\infty} \frac{1}{2N}\left\langle \ln \operatorname{Tr} \prod_{i=1}^{N} M(\varepsilon_i)\left(\prod_{i=1}^{N} M(\varepsilon_i)\right)^{+}\right\rangle \tag{3}$$

where the angle bracket denotes an average over the realisations of the noise (the ε_i are independent).

For the example of the discretised 1D Schrödinger equation in a random potential, one has ($n=2$)

$$\begin{bmatrix} \psi_{i+1} \\ \psi_i \end{bmatrix} = \begin{bmatrix} 2-\varepsilon_i & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \psi_i \\ \psi_{i-1} \end{bmatrix}. \tag{4}$$

Consider now the linear mapping operating in the space of $n \times n$ matrices:

$$\varphi_i : u \to \varphi_i(u) = M(\varepsilon_i)uM^{+}(\varepsilon_i). \tag{5}$$

This transformation can be represented by the $n^2 \times n^2$ matrix $\varphi_{\varepsilon_i} = M(\varepsilon_i) \otimes M(\varepsilon_i)$ where $\otimes$ denotes the usual tensorial product. λ can be rewritten in term of φ_{ε_i}:

$$\lambda = \lim_{N\to\infty} \frac{1}{2N}\left\langle \ln \operatorname{Tr}_{n^2}\left(\prod_{i=1}^{N} \varphi_{\varepsilon_i}\right)\mathbb{1}_{n^2}\right\rangle. \tag{6}$$

Using the convexity of the logarithm, one can give an upper bound on λ:

$$\lambda \leqslant \lim_{N\to\infty} \frac{1}{2N} \ln \operatorname{Tr}_{n^2}\left(\prod_{i=1}^{N} \langle\varphi_{\varepsilon_i}\rangle \mathbb{1}_{n^2}\right) = \tfrac{1}{2}\ln \Lambda_{\max} \tag{7}$$

where $\Lambda_{\max}$ is the largest eigenvalue of the *averaged* $n^2 \times n^2$ matrix $\langle\varphi_\varepsilon\rangle = \langle M(\varepsilon) \otimes M(\varepsilon)\rangle$. (The last equality in (7) holds provided $\mathbb{1}_{n^2}$ has a component in the eigenspace corresponding to $\Lambda_{\max}$, which is the generic case.)

The bound (7) thus amounts to replacing the original disordered problem by an effective pure one: in this sense, it can be called an annealed method. Note however that the dimension of the corresponding matrices has been increased from $n \times n$ to $n^2 \times n^2$.

For the localisation problem (4), one has to compute the largest eigenvalue of the following 4×4 matrix:

$$\begin{bmatrix} (2-\mu)^2+\sigma & \mu-2 & \mu-2 & 1 \\ 2-\mu & 0 & -1 & 0 \\ 2-\mu & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

whose characteristic polynomial is

$$x[-x^3+(\sigma+\mu^2-4\mu)x^2+(3\sigma+\mu^2-4\mu)x+2\sigma]=0 \tag{9}$$

where $\Lambda=1+x$.

From (9), we obtain the following results.

(i) At band edge ($\mu=0$ or 4) the LE $\lambda(\mu=0,\sigma)$ is bounded from above by $2^{-2/3}\sigma^{1/3}\simeq 0.63\sigma^{1/3}$, to be compared with the exact result $0.289\ldots\sigma^{1/3}$ (Derrida and Gardner 1984).

(ii) In the vicinity of the band edge, the upper bound on $\lambda/\sigma^{1/3}$ given by (9) depends, for small σ, on the scaling variable $\mu/\sigma^{2/3}$, as is the case for the exact result. This bound is plotted in figure 1 together with the result of Derrida and Gardner. As expected, these two curves are very close far from the critical value $\mu=0$.

(iii) Away from the band edge, (9) leads to

$$\lambda(\mu,\sigma)\leqslant\frac{\sigma}{\mu(4-\mu)}$$

which (see equation (14) below) is just twice the exact value known from weak disorder expansions (forgetting the values of μ where an anomaly arises (Kappus and Wegner 1981): $\mu=2(1+\cos\pi n/m)$). (In figure 1, we also give numerical results concerning the imaginary noise case, for which the bound is surprisingly nearly exact in a wide range of $\mu/\sigma^{2/3}$.)

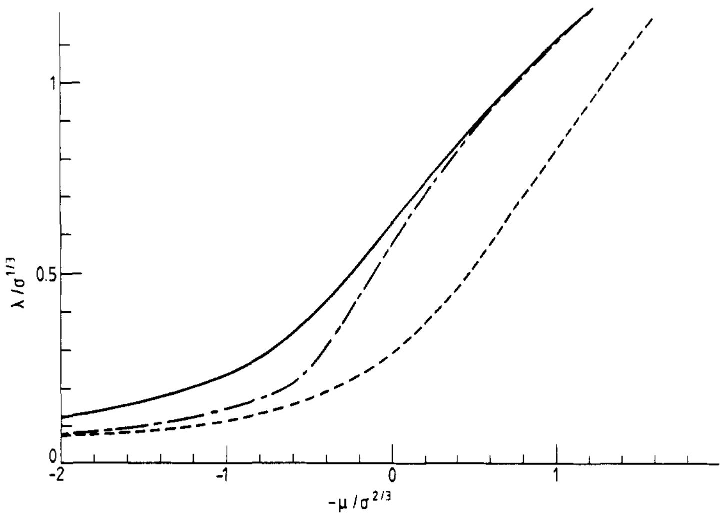

Figure 1. The ratio $\lambda/\sigma^{1/3}$ as a function of the scaling variable $\mu/\sigma^{2/3}$, near the band edge for small σ. Full curve: upper bound; broken curve: imaginary noise, dotted curve: real noise.

Let us also mention that these bounds can be improved using a method detailed in another paper on the example of 2D spin systems (Georges *et al* 1986). It is based on the following trick: let $f(\varepsilon)$ be an arbitrary positive function of ε; one has

$$\langle \ln \mathrm{Tr}\varphi_\varepsilon\rangle \leqslant \langle \ln 1/f\rangle + \ln\langle f\, \mathrm{Tr}\, \varphi_\varepsilon\rangle.$$

One can now optimise the bound by minimising functionally the right-hand side with respect to f.

At the band edge ($\mu \to 0$) and for $\sigma \to 0$, it can be seen that the minimum is obtained for

$$f(\varepsilon) = \frac{1}{1 + b\sigma^{-1/3}\varepsilon} \qquad b \simeq 0.665$$

which leads to the upper bound $\lambda \leqslant 0.435\sigma^{1/3}$.

This can also be systematically done in the vicinity of the band edge, leading to a function of $\mu/\sigma^{2/3}$ lying halfway between the two curves of figure 1.

Let us now turn to problem described by (1) for arbitrary n, with a zero mean value for $\varepsilon(x)$. The discretised version of (1) is (up to irrelevant terms in the weak noise limit)

$$\boldsymbol{V}_{i+1} = \boldsymbol{M}_i \boldsymbol{V}_i$$

where M_i is the $n \times n$ matrix

$$\boldsymbol{M}_i = \begin{bmatrix} 1 & 0 & \cdots & \varepsilon_i \\ 1 & 1 & 0 & 0 \\ \vdots & 1 & \ddots & \vdots \\ 1 & \cdots & & 1 \end{bmatrix} \tag{10}$$

and $\boldsymbol{V}_i = (\Delta^{n-1}\psi_i, \ldots, \Delta^0\psi_i)$; Δ^i is the finite difference operator and $\langle \varepsilon_i \varepsilon_j\rangle = \sigma\delta_{ij}$.

The average mapping $\langle \varphi_{\varepsilon_i}\rangle$ is now represented by the $n^2 \times n^2$ matrix $\langle \boldsymbol{M}_i \otimes \boldsymbol{M}_i\rangle$ whose characteristic polynomial can be evaluated using the algebra of the following $n \times n$ matrices:

$$(A_k)_{ij} = \delta_{i,j-k} \qquad 1 \leqslant i, j, k \leqslant n.$$

The rules of this algebra are given by

$$\begin{aligned} A_k A_p &= A_{k+p} && \text{if } k + p \leqslant n \\ &= 0 && \text{if not.} \end{aligned}$$

This allows us to compute the matrix $\ln(\mathbb{1}_{n^2} - \lambda\langle\varphi\rangle)$ and the characteristic polynomial is obtained by using the formula $\det M = \exp \mathrm{Tr} \ln M$. Finally, it turns out that the largest root $\Lambda_{\max}$ of this polynomial of degree n^2 satisfies the following equation:

$$0 = 1 - \sigma \sum_{k=0}^{\infty} \Lambda^k \Bigg\{ \sum_{\substack{p_1 + p_2 + \ldots p_d = k \\ p_1 + 2p_2 + \ldots + dp_d = (k-1)d+1}} \frac{k!}{p_1! \ldots p_d!} \Bigg\}^2 + \text{irrelevant terms.}$$

The highest pole in $(1 - \Lambda)$ and its residue can easily be extracted from this series, and this leads to the bound (2) when $\sigma \to 0$. The $1/(2n-1)$ exponent is the correct one, as can be intuitively understood by noting that the tensorial product in (6) acts as a first step of a block renormalisation analysis.

We have numerically studied the relevance of this bound: the ratio of the simulated LE on the 'annealed bound' is plotted in figure 2 as a function of n.

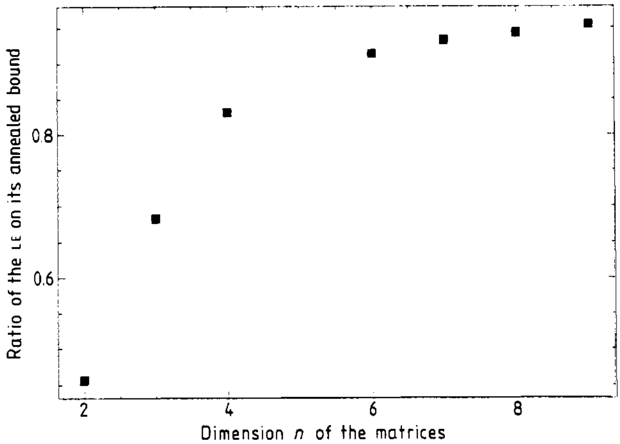

Figure 2. The ratio of the numerically computed LE on the annealed bound (2) as a function of the size of the matrix n.

This suggests that (2) might be the *exact* analytical expression of $\lambda(\sigma\to 0)$ in the limit $n\to\infty$. The exact prefactor of $\sigma^{1/(2n-1)}$ would require one to solve a partial differential equation of $n-1$ variables generalising the simple differential equation (32) of Derrida and Gardner (1984).

The above bound can in fact be seen as the second replica moment of $\mathrm{Tr}\,\Pi\, M(\varepsilon_i)$. More generally, in order to obtain information on the entire probability distribution of $\mathrm{Tr}\,\Pi\, M(\varepsilon_i)$, it is possible to study

$$L(q, N)=\frac{1}{N}\log\left\langle\left(\mathrm{Tr}\prod_{i=1}^{N} M(\varepsilon_i)\right)^q\right\rangle$$

(the LE is then $\lim_{q\to 0}(L(q,\infty)/q)$). Noting that

$$\left\langle\left(\mathrm{Tr}_2\prod_{i=1}^{N} M_{\varepsilon_i}\right)^q\right\rangle=\mathrm{Tr}_{2^q}\left\langle\prod_i M_{\varepsilon_i}^{\otimes q}\right\rangle=\mathrm{Tr}_{2^q}\langle M_\varepsilon^{\otimes q}\rangle^N \tag{11}$$

one has to estimate the highest eigenvalues Λ_q of a $2^q\times 2^q$ matrix $\langle M_\varepsilon^{\otimes q}\rangle$, which can in fact be reduced to a $(q+1)\times(q+1)$ matrix using symmetry properties (Pendry 1982).

Actual calculations can be performed in the weak disorder limit within the framework of standard perturbation theory. This method has been applied by Kirkman and Pendry (1984) to study analytically the moments of an analogous quantity, namely the transmission coefficient, away from the band centre ($\mu=2$) where the perturbation theory becomes degenerate and where only a semi-numerical treatment of the diagonalisation problem was performed.

Let us notice that for the moments of Tr M, one has a nice interpretation in terms of tensorial products of q $\frac{1}{2}$ spins, which greatly simplifies actual calculations and allows a complete analytical treatment of the degenerate case (band centre). Denoting by $\boldsymbol{S}$ the total spin operator, one has, in every eigenspace of S_z,

$$\langle M_\varepsilon^{\otimes q}\rangle=\exp\left(2\mathrm{i}\theta S_z+\frac{\sigma}{\mu(4-\mu)}(S^2-3S_z^2)\right)\qquad \text{for } 2\cos\theta=2-\mu\neq 0 \tag{12}$$

and

$$\langle M_\varepsilon^{\otimes q}\rangle=\exp[\mathrm{i}\pi S_z+\tfrac{1}{2}\sigma(S_y^2-S_z^2]\qquad \text{for } \mu=2. \tag{13}$$

For $\mu \neq 2$, $L(q, N)$ can be evaluated for any finite N and analytically continued to any real q. This allows us to discuss the influence of the lattice in the metallic (finite-size) regime and will be addressed in a subsequent work.

In the thermodynamic limit $N \to \infty$, one obtains

$$L(q, \infty) = \frac{\sigma}{\mu(4-\mu)} \max_{\substack{S=q/2, q/2-1, \ldots \\ |m| \leq S}} \{(S(S+1)) - 3m^2\} = \lambda(q + \tfrac{1}{2}q^2) \qquad \text{for } q \text{ even} \tag{14}$$

allowing one to analytically continue for any real q the expression of the average modulus of the trace $\langle|\Pi\ M|^q\rangle$, showing that the distribution of this quantity is purely log-normal with LE $\lambda = \sigma/2\mu(4-\mu)$. Analogous results have first been obtained by Melnikov (1981a) in the context of the random phase model (Anderson *et al* 1980) and also by Kirkman and Pendry (1984).

In the interesting band centre case ($\mu = 2$), the spectrum of operator (13) for an integer q is given by the roots of two sets of polynomials $P_S^a(X)$ and $P_S^b(X)$, corresponding to the representation of total spin S (running from 0 (or $\frac{1}{2}$) to $q/2$, the maximally symmetrical representation). These $P_S^{a,b}$ obey the same recursion relations with different initial conditions:

$$Q_S^m(X) = (a_{Sm} - X)Q_S^{m-2}(X) - C_{S+m}^2 C_{S-m+2}^2 Q_S^{m-4}(X) \tag{15a}$$

with

$$a_{Sm} = S(S+1) - 3m^2$$

and, for integer S,

$$\begin{aligned} &P_S^a = Q_S^S \qquad &Q_S^{-S-2}(X) = 1 \qquad &Q_S^{-S}(X) = a_{S,-S} - X \\ &P_S^b = Q_S^{S-1} \qquad &Q_S^{-S-1}(X) = 1 \qquad &Q_S^{-S+1}(X) = a_{S,-S+1} - X \end{aligned} \tag{15b}$$

while for half-integer S,

$$P_S^a = P_S^b = Q_S^{S-1} \qquad Q_S^{-S-2}(X) = 1 \qquad Q_S^{-S}(X) = a_{S,-S} - X. \tag{15c}$$

The recursion relation naturally closes for q integer and the analytic continuation simply corresponds to the iteration of (15a) *ad infinitum*. This introduces spurious roots which turn out to be always smaller than the natural one.

Taking the thermodynamic limit amounts to selecting the largest root which is then easy to follow for real q, as seen in figure 3 which represents $L(q, \infty)/q$ at the band centre. This figure calls for three remarks.

(a) The $q \to 0$ limit gives the exact Kappus-Wegner value of the LE as will be explained below. Thus no difficulty arises when taking the thermodynamic limit *before* the $q = 0$ limit. This is true for all energies within the band, and is due to the fact that eigenvalues do not accumulate in this limit (see figure 3).

(b) Melnikov (1981b), by studying a simplified version of the model, has obtained the correct asymptotic ($q \to \infty$) slope of $L(q)/q$.

(c) For negative q, $L(q)/q$ does not seem to exhibit any peculiarities, such as the annulation of $L(-2)$, as suggested by Melnikov (1981b).

One can obtain the exact value for the LE λ by developing the recursion (15a) for small q. The λ term turns out to be the limiting ratio of two sequences satisfying

$$u_{n+2} = -12n^2 u_{n+1} - n(2n-1)^2(n-1)u_n \tag{16a}$$

$$\lambda = \frac{1}{8} \frac{u_\infty(u_2 = -1, u_3 = 11)}{u_\infty(u_2 = -1, u_3 = 12)}. \tag{16b}$$

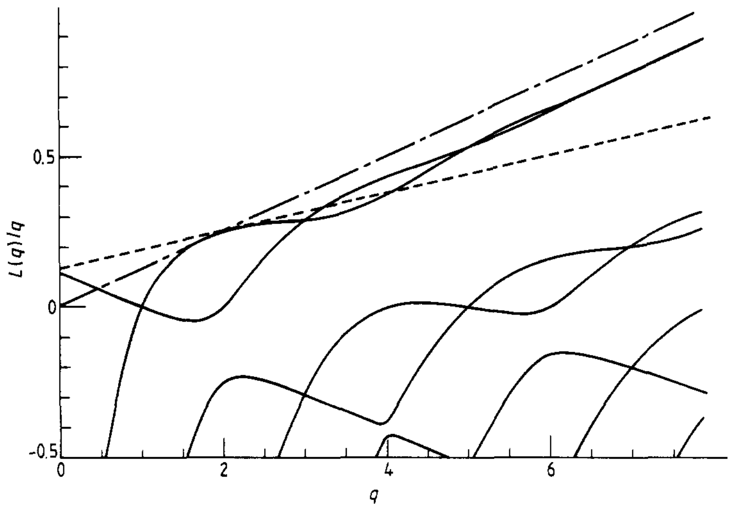

Figure 3. $L(q)/q$ for the band centre case. Plain lines: the roots of polynomials P_a and P_b as functions of q. The value of $L(q)/q$ is the largest root for a given q (note the occurrence of spurious roots). Chain curve: Melnikov's estimation; dotted line: continuation to the case $\mu = 2$ of the result valid only for $\mu \neq 2$ (without anomaly) for the modulus of the trace.

This ratio converges extremely rapidly towards the exact value $\lambda = 0.1142$ (Derrida and Gardner 1984). Furthermore, a generating function treatment of (16*a*) allows us to exactly obtain

$$\lambda = \frac{1}{8}\frac{11-R}{12-R}$$

with

$$R = \frac{I(\frac{5}{4}, \frac{7}{4}, 3)}{I(\frac{3}{4}, \frac{5}{4}, 2)} = 0.383\ldots$$

and

$$I(a, b, c) = \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1-\frac{t}{9}\right)^{-a} \mathrm{d}t.$$

The complete distribution law for the band centre, which is no longer log-normal, can, in principle, be reconstructed from the knowledge of $L(q)$ in the thermodynamic limit.

In the case $\mu = 0$ (band edge), no perturbation theory has yet been applied. We have nonetheless shown that for q integer $\geqslant 2$, the $\sigma^{1/3}$ behaviour always holds for weak disorder. The $q = 2$ and 4 replica moments yield

$$\frac{\Lambda_q - 1}{q\sigma^{1/3}} = \frac{2^{1/3}}{2}, \frac{42^{1/3}}{4}$$

and a linear extrapolation gives $\lambda \simeq 0.39\sigma^{1/3}$ while the exact value is $0.289\sigma^{1/3}$. If however one includes the third moment ($q = 3$) $\langle \mathrm{Tr}(\Pi_{i=1}^{N} M_i)^3 \rangle$, one obtains the intermediate value $(12)^{1/3}/3$ and obtains $\lambda \simeq 0.283\sigma^{1/3}$, again very close to the true value.

Let us summarise the results contained in this letter. They include

(i) a systematic way of obtaining rigorous upper bounds on the LE of products of random matrices,

(ii) an upper bound on the LE of the first problem (i), leading to a conjecture on its exact behaviour in the limit $n \to \infty$, and

(iii) a study of the moments of $\mathrm{Tr}\,\Pi\, M$ in the infinite-size limit showing that the statistics at the band centre is far more complicated than the log-normal distribution holding within the band.

Similar considerations have led us to improvements of the annealed approaches for 2D disordered spin systems (Georges *et al* 1986), which provide approximations of the critical lines in good agreement with numerical results. Let us emphasise that these methods apply beyond the weak disorder limit and can also be used in the case of complex random matrices.

The evolution of the statistics with respect to the size of the sample will be the object of a subsequent publication.

We thank B Derrida and R Rammal for their encouragment.

References

Anderson P W, Thouless D J, Abrahams E and Fisher D 1980 *Phys. Rev.* B **22** 350
Bouchaud J P and Le Doussal P 1986 *J. Phys. A: Math. Gen.* **19** 797
Derrida B and Gardner E 1984 *J. Physique* **45** 1283
Georges A, Hansel D, Le Doussal P, Maillard J M and Bouchard J P 1986 *J. Physique* **47** 947
Halperin B I 1965 *Phys. Rev.* **139** A104
Ishii K 1973 *Prog. Theor. Phys. Suppl.* **53** 77
Kappus M and Wegner E 1981 *Z. Phys.* B **45** 15
Kirkman P S and Pendry J B 1984 *J. Phys. C: Solid State Phys.* **17** 5707, 4327
Melnikov V I 1981a *Sov. Phys.-Solid State* **23** 444
—— 1981b *JETP Lett.* **34** 451
Pendry J B 1982 *J. Phys. C: Solid State Phys.* **15** 4821
Thouless D J 1972 *J. Phys. C: Solid State Phys.* **5** 77
—— 1974 *Phys. Rep.* **13** 93